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Fixed points of k-nonexpansive mappings

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Abstract. In this paper, we generalizes the notion of fundamentally nonexpansive mappings introduced in [Fixed Point Theory and Applications, 2014 (2014), Paper No. 76] to k-nonexpansive mappings. We investigate various properties associated with these mappings and demonstrate some existence and convergence results for k-nonexpansive mappings in the setting of Banach spaces.

1. Introduction

Nonexpansive mappings are those where the distance between mapped points does not increase. Consequently, these mappings exhibit uniform continuity within their domain for instances isometries, contractions, and resolvent of accretive operators. In 1965, Kirk [10], Browder [2], and Göhde [6] independently established a fixed point theorem for nonexpansive mappings defined on a closed, bounded, convex subset of a uniformly convex Banach space. Since then, the fixed point theory for nonexpansive mappings has developed into a vibrant field of research. Many researchers have extended and generalized the notion of nonexpansiveness to more general conditions. In this sequel, in 2008, Suzuki [19] refined the notion of nonexpansiveness to the condition (C) (see Definition 2). This condition (C) is noteworthy because it does not require to hold throughout the domain of the mappings and does not necessitate continuity within their domain. For a thorough exploration of the generalizations of nonexpansive mappings, we refer to [1, 2, 4, 9, 10, 12–18].

Another important generalization of nonexpansive mappings was proposed by Hosseini Ghoncheh and Razani [7] in 2014. They introduced the notion of fundamentally nonexpansive (FNE, for short) mappings which includes the class of nonexpansive mappings and those satisfying the condition (C). In 2016, Moosaei [11] explored some basic properties of these mappings and established some existence and convergence results for these mappings

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under the premise that the image set of such mappings is bounded and convex. However, in 2018, Domínguez Benavides, and Llorens-Fuster [3] raised the question of whether this class of mappings has an approximate fixed point sequence (afps) on a nonempty closed convex and bounded subset of Banach spaces.

Motivated by the above, in this paper, we generalize the notion of FNE mappings to k-nonexpansive mappings. We demonstrate the class of k-nonexpansive mappings includes several classes of mappings including nonexpansive mappings, FNE mapping, those satisfying the condition (C) and others. We establish that the set of fixed points for a k-nonexpansive mapping defined on a nonempty convex subset of a strictly convex normed space is also convex. Additionally, we prove some results that guarantee the existence of an afps for k-nonexpansive mappings under the specific range of the mappings. Finally, we prove a fixed point result for k-nonexpansive mapping in the setting of Banach space with a normal structure, which provides a partial answer to the question regarding FNE mappings.

2. Preliminaries

Throughout this note, the set of all positive integers is denoted by $\mathbb{N}$, the open ball with center ω and radius r by $B(\omega, r)$ and the closure of W by $\mathrm{cl}(W)$. We will assume that W is a nonempty subset of a Banach space $(Y, \|\cdot\|)$.

A Banach space Y is defined as uniformly convex if, for every $\varepsilon \in (0,2]$, there exists a $\delta > 0$ such that for any $\omega, \vartheta \in Y$ satisfying $\|\omega\| \leq 1$, $\|\vartheta\| \leq 1$, and $\|\omega - \vartheta\| \geq \varepsilon$, it follows that $\left\|\frac{\omega+\vartheta}{2}\right\| \leq 1 - \delta$. The space Y is referred to as uniformly convex in every direction (UCED) if, for $\varepsilon \in (0,2]$ and any unit vector $\mu \in Y$, there exists a $\delta(\varepsilon, \mu) > 0$ such that for all $\omega, \vartheta \in Y$ with $\|\omega\| \leq 1$, $\|\vartheta\| \leq 1$, and $\omega - \vartheta \in \{t\mu : t \in [-2, -\varepsilon] \cup [\varepsilon, 2]\}$, we have $\left\|\frac{\omega+\vartheta}{2}\right\| \leq 1 - \delta(\varepsilon, \mu)$. Furthermore, a Banach space Y has the Opial property if for any sequence $\{\omega_n\} \subset Y$ that converges weakly to a point $\omega \in Y$, it holds that $\liminf_{n\to\infty} \|\omega_n - \omega\| < \liminf_{n\to\infty} \|\omega_n - \vartheta\|$ for all $\vartheta \in Y$ with $\vartheta \neq \omega$. For each element ω in W, we define $r_\omega(W) = \sup_{\vartheta \in W} \|\omega - \vartheta\|$. A point $\omega \in W$ is defined as non-diametral if $r_\omega(W) < \mathrm{diam}(W)$, where $\mathrm{diam}(W)$ indicates the diameter of W. A convex subset W of Y is said to have a normal structure if every bounded, convex subset $C \subset W$ with $\mathrm{diam}(C) > 0$ contains a non-diametral point. Similarly, a Banach space Y is said to have a normal structure if this property holds for every convex subset of Y. It is important to note that uniformly convex Banach spaces inherently have a normal structure. However, the Opial property does not directly imply a normal structure.

Definition 1. Let Γ be a self-mapping defined on W. We say that the mapping Γ is nonexpansive if $\|\Gamma(\omega) - \Gamma(\vartheta)\| \leq \|\omega - \vartheta\|$ for every pair $\omega, \vartheta \in W$.

We denote $F(\Gamma)$ as the set of fixed points of Γ, which is given by $F(\Gamma) = \{\omega \in W : \Gamma(\omega) = \omega\}$. The mapping Γ is referred to as quasi-nonexpansive (QNE, for short) if its fixed point set is nonempty and it satisfies the property $\|\Gamma(\omega) - u\| \leq \|\omega - u\|$ for all $\omega \in W$ and $u \in F(\Gamma)$.

Definition 2 ([19]). A mapping $\Gamma : W \to W$ is said to satisfy the condition (C) if for all $\omega, \vartheta \in W$,

$$\frac{1}{2}\|\omega - \Gamma(\omega)\| \leq \|\omega - \vartheta\| \implies \|\Gamma(\omega) - \Gamma(\vartheta)\| \leq \|\omega - \vartheta\|.$$

The following definition is introduced by Hosseini Ghoncheh and Razani in [7].

Definition 3. Let W be a nonempty subset of a normed space $(Y, \|\cdot\|)$. A mapping $\Gamma : W \to Y$ is said to be FNE if it satisfies the following condition for each $\omega, \vartheta \in W$:

$$\|\Gamma^2(\omega) - \Gamma(\vartheta)\| \leq \|\Gamma(\omega) - \vartheta\|.$$

It is clear from the definition above that the class of FNE mappings contains is wider than that of nonexpansive mappings and mappings satisfying the condition (C), however the class of FNE mappings is different from those satisfying the condition (C). The following example presented in [11, Example 1.2] reveals this point.

Example 1. Define a mapping Γ on $[0, 4]$ as follows:

$$\Gamma(\omega) = \begin{cases} 1, & \omega \neq 4; \\ 2.5, & \omega = 4, \end{cases}$$

for every $\omega \in [0, 4]$. In this case, Γ is a FNE, yet it does not meet condition (C), implying that Γ is not nonexpansive.

Lemma 1 ([5]). *Consider two bounded sequences $\{\omega_n\}$ and $\{\vartheta_n\}$ in a Banach space Y, where $\omega_{n+1} = \lambda\vartheta_n + (1 - \lambda)\omega_n$ and it holds that $\|\vartheta_n - \vartheta_{n+1}\| \leq \|\omega_n - \omega_{n+1}\|$ for every $n \in \mathbb{N}$, with $\lambda \in (0, 1)$. Then it follows that* $\lim_{n\to\infty} \|\omega_n - \vartheta_n\| = 0$.

Definition 4 ([8]). A mapping $\Gamma : W \to W$ is said to satisfy the condition (E_μ) on W, if there exists $\mu \geq 1$ such that for all $\omega, \vartheta \in W$

$$\|\omega - \Gamma(\vartheta)\| \leq \mu\|\omega - \Gamma(\omega)\| + \|\omega - \vartheta\|.$$

3. k-Nonexpansive mappings

We now extend the concept of fundamentally nonexpansive (FNE) mappings to a more general category known as k-nonexpansive mappings, defined as follows.

Definition 5. Let W be a nonempty subset of a normed linear space Y. A mapping $\Gamma : W \to W$ is called a k-nonexpansive mapping if there exists a positive integer $k \in \mathbb{N}$ such that

$$\|\Gamma^k(\omega) - \Gamma(\vartheta)\| \leq \|\Gamma^{k-1}(\omega) - \vartheta\| \tag{1}$$

holds for all $\omega, \vartheta \in W$.

From this definition, it is evident that the class of k-nonexpansive mappings reduces to nonexpansive mapping for $k = 1$, and to FNE mappings for $k = 2$. However, for values of $k \geq 3$, k-nonexpansive mappings may different from FNE mappings and do not necessarily satisfy condition (C) in certain situations. The following example demonstrates this distinction.

Example 2. Consider the mapping $\Gamma : [0, 4] \to [0, 4]$ defined as:

$$\Gamma(\omega) = \begin{cases} 0.5, & \text{if } 0 \leq \omega < 2; \\ 1.5, & \text{if } 2 \leq \omega < 4; \\ 2.5, & \text{if } \omega = 4. \end{cases}$$

Clearly, $\Gamma^3(\omega) = 0.5$ for all $\omega \in [0, 4]$, and we can verify that

$$\left|\Gamma^3(\omega) - \Gamma(\vartheta)\right| \leq \left|\Gamma^2(\omega) - \vartheta\right|$$

holds for all $\omega, \vartheta \in [0, 3]$. Therefore, Γ qualifies as a 3-nonexpansive mapping. However, for $\omega = 4$ and $\vartheta = 1.9$, we find that

$$\left|\Gamma^2(\omega) - \Gamma(\vartheta)\right| = \left|1.5 - 0.5\right| = 1 > 0.6 = \left|2.5 - 1.9\right| = \left|\Gamma(\omega) - \vartheta\right|,$$

and similarly, for $\omega = 2$ and $\vartheta = 1.9$, it can be verified that $|\Gamma(\omega) - \Gamma(\vartheta)| > |\omega - \vartheta|$. Hence, Γ is neither an FNE mapping nor a nonexpansive mapping on the interval $[0, 4]$.

Remark 1. It is worth noting that every k-nonexpansive mapping is also $(k + 1)$-nonexpansive; however, the reverse does not hold (see Examples 1 and 2).

Lemma 2. *Let W be a nonempty subset of a normed space Y, and let $\Gamma : W \to W$ be a k-nonexpansive mapping. Then, for any elements $\omega, \vartheta \in W$, the following inequality is satisfied:*

$$\left\|\omega - \Gamma(\vartheta)\right\| \leq \left\|\omega - \Gamma(\omega)\right\| + 2\left\|\Gamma^{k-1}(\omega) - \omega\right\| + \left\|\omega - \vartheta\right\|.$$

Proof. By setting $\omega = \vartheta$ in (1), it is straightforward to verify that $\|\Gamma^k(\omega) - \Gamma(\omega)\| \leq \|\Gamma^{k-1}(\omega) - \omega\|$ holds for all $\omega \in W$. For $\omega, \vartheta \in W$ where $\omega \neq \vartheta$, we obtain the following:

$$\begin{aligned} \left\|\omega - \Gamma(\vartheta)\right\| &\leq \left\|\omega - \Gamma(\omega)\right\| + \left\|\Gamma^k(\omega) - \Gamma(\omega)\right\| + \left\|\Gamma^k(\omega) - \Gamma(\vartheta)\right\| \\ &\leq \left\|\omega - \Gamma(\omega)\right\| + \left\|\Gamma^{k-1}(\omega) - \omega\right\| + \left\|\Gamma^{k-1}(\omega) - \vartheta\right\| \\ &\leq \left\|\omega - \Gamma(\omega)\right\| + 2\left\|\Gamma^{k-1}(\omega) - \omega\right\| + \left\|\omega - \vartheta\right\|. \end{aligned}$$

□

The following lemma can be easily proven.

Lemma 3. *Let W be a nonempty subset of a normed space Y, and let $\Gamma : W \to Y$ be a mapping. The following properties are observed:*

(i) *If Γ is nonexpansive, then Γ is k-nonexpansive.*
(ii) *If Γ is k-nonexpansive and the set of fixed points $F(\Gamma)$ is nonempty, then Γ is quasi-nonexpansive (QNE).*

Lemma 4. *Let $\Gamma : W \to W$ be a k-nonexpansive mapping, where W is a nonempty subset of a Banach space Y. The set $F(\Gamma)$ is closed. Furthermore, if Y is strictly convex and W is convex, then $F(\Gamma)$ is also convex.*

Proof. Suppose, for the sake of contradiction, that there exists a point ω in $\mathrm{cl}(F(\Gamma))$ such that $\omega \notin F(\Gamma)$. Set $r = \|\omega - \Gamma(\omega)\|/3$. Since $\omega \in \mathrm{cl}(F(\Gamma))$, the intersection $B(\omega, r) \cap F(\Gamma)$ must be nonempty. Choose $u \in B(\omega, r) \cap F(\Gamma)$, thus $\|\omega - u\| < r$ and $\Gamma(u) = u$. By Lemma 3 (ii), we have:

$$\begin{aligned}\big\|\omega - \Gamma(\omega)\big\| &\leq \big\|\omega - u\big\| + \big\|\Gamma(\omega) - u\big\| \\ &\leq 2\big\|\omega - u\big\| \\ &< 2r \\ &= \frac{2}{3}\big\|\omega - \Gamma(\omega)\big\|,\end{aligned}$$

leading to a contradiction, therefore proving that $F(\Gamma)$ is closed.

Next, assume Y is strictly convex and W is convex; we aim to show $F(\Gamma)$ is convex. Let $\lambda \in (0, 1)$ and take $\omega, \vartheta \in F(\Gamma)$ with $\omega \neq \vartheta$. Define $u = \lambda\omega + (1 - \lambda)\vartheta$. By Lemma 3, we find

$$\begin{aligned}\big\|\omega - \vartheta\big\| &\leq \big\|\Gamma(u) - \omega\big\| + \big\|\Gamma(u) - \vartheta\big\| \\ &\leq \big\|u - \omega\big\| + \big\|u - \vartheta\big\| = \|\omega - \vartheta\|.\end{aligned}$$

Thus,

$$\begin{aligned}\big\|\omega - \vartheta\big\| &= \big\|\Gamma(u) - \omega\big\| + \big\|\Gamma(u) - \vartheta\big\| \\ &= \big\|u - \omega\big\| + \big\|u - \vartheta\big\|.\end{aligned}$$

Since Y is strictly convex, there exists $t \in [0, 1]$ such that $\Gamma(u) = t\omega + (1 - t)\vartheta$. Moreover,

$$\begin{aligned}(1 - t)\big\|\omega - \vartheta\big\| &= \big\|\Gamma(\omega) - \Gamma(u)\big\| \\ &\leq \big\|\omega - u\big\| = (1 - \lambda)\big\|\omega - \vartheta\big\|,\end{aligned}$$

and

$$\begin{aligned}t\big\|\omega - \vartheta\big\| &= \big\|\Gamma(\vartheta) - \Gamma(u)\big\| \\ \leq \big\|\vartheta - u\big\| &= \lambda\big\|\omega - \vartheta\big\|.\end{aligned}$$

This implies $1 - t \leq 1 - \lambda$ and $t \leq \lambda$, leading to $\lambda = t$.

Thus, we conclude that $u \in F(\Gamma)$. □

4. CONVERGENCE AND EXISTENCE RESULTS FOR k-NONEXPANSIVE MAPPINGS

Firstly, we will prove some results that guarantee the existence of afps for k-nonexpansive mappings in the setting of Banach spaces under certain conditions.

Lemma 5. *Let $\Gamma : W \to W$ be a k-nonexpansive mapping, where W is a nonempty subset of a Banach space Y, and let the set $\Gamma^{k-1}(W)$ be bounded and convex. We define a sequence $\{\Gamma^{k-1}(\omega_n)\}$ in $\Gamma^{k-1}(W)$ by starting with $\omega_1 \in W$ and applying the following formula:*

$$\Gamma^{k-1}(\omega_{n+1}) = \lambda\Gamma^k(\omega_n) + (1-\lambda)\Gamma^{k-1}(\omega_n), \tag{2}$$

for all $n \in \mathbb{N}$ and for any $\lambda \in (0,1)$. Then, $\{\Gamma^{k-1}(\omega_n)\}$ is an afps for Γ.

Proof. Given that Γ is a k-nonexpansive mapping, it holds that

$$\big\|\Gamma^k(\omega_{n+1}) - \Gamma^k(\omega_n)\big\| \leq \big\|\Gamma^{k-1}(\omega_{n+1}) - \Gamma^{k-1}(\omega_n)\big\|, \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 1, it follows that

$$\lim_{n\to\infty}\big\|\Gamma^{k-1}(\omega_n) - \Gamma^k(\omega_n)\big\| = 0.$$

This concludes the proof of the lemma. □

Theorem 1. *Let W be a nonempty compact subset of a Banach space Y. Suppose that $\Gamma : W \to W$ is a k-nonexpansive mapping, and that $\Gamma^{k-1}(W)$ is convex. Under these conditions, the sequence $\{\Gamma^{k-1}(\omega_n)\}$ defined in (2) converges strongly to a fixed point of Γ.*

Proof. As W is compact, there exists a subsequence $\{\Gamma^{k-1}(\omega_{n_j})\}$ of $\{\Gamma^{k-1}(\omega_n)\}$ such that $\lim_{j\to\infty}\Gamma^{k-1}(\omega_{n_j}) = u$ for some $u \in W$. Using (2), we get

$$\lim_{j\to\infty}\Gamma^k(\omega_{n_j}) = u.$$

Since Γ is a k-nonexpansive, we have

$$\big\|\Gamma^{k-1}(\omega_{n_j}) - \Gamma(u)\big\| \leq \big\|\Gamma^{k-1}(\omega_{n_j}) - u\big\|, \quad \text{for all } j \in \mathbb{N}.$$

This suggests that

$$\lim_{j\to\infty}\Gamma^k(\omega_{n_j}) = \Gamma(u).$$

Hence, u is a fixed point of Γ. Conversely, we have

$$\begin{aligned}\big\|\Gamma^{k-1}(\omega_{n+1}) - u\big\| &= \big\|\lambda\Gamma^k(\omega_n) + (1-\lambda)\Gamma^{k-1}(\omega_n) - \Gamma(u)\big\| \\ &\leq \lambda\big\|\Gamma^k(\omega_n) - \Gamma(u)\big\| + (1-\lambda)\big\|\Gamma^{k-1}(\omega_n) - u\big\| \\ &\leq \big\|\Gamma^{k-1}(\omega_n) - u\big\|, \quad \text{for all } n \in \mathbb{N}.\end{aligned}$$

The sequence $\{\|\Gamma^{k-1}(\omega_n) - u\|\}$ is bounded and decreasing, hence it is convergent. As $\lim_{j\to\infty} \Gamma^{k-1}(\omega_{n_j}) = u$, we conclude that

$$\lim_{n\to\infty} \Gamma^{k-1}(\omega_n) = u. \qquad \square$$

Theorem 2. *Let W be a nonempty, weakly compact subset of a Banach space Y that satisfies the Opial property. Suppose $\Gamma : W \to W$ is k-nonexpansive, and the set $\Gamma^{k-1}(W)$ is convex. Then the sequence $\{\Gamma^{k-1}(\omega_n)\}$ generated by (2) converges weakly to a fixed point of Γ.*

Proof. Since W is weakly compact, the sequence $\{\Gamma^{k-1}(\omega_n)\}$ has a subsequence $\{\Gamma^{k-1}(\omega_{n_l})\}$ that converges weakly to some point $u \in W$. We aim to show that u is a fixed point of Γ. Suppose, contrary to our claim, that $\Gamma(u) \neq u$. By Lemma 1, for all $l \in \mathbb{N}$, it follows that

$$\big\|\Gamma^{k-1}(\omega_{n_l}) - \Gamma(u)\big\| \leq (2k-1)\big\|\Gamma^{k-1}(\omega_{n_l}) - \Gamma^{k}(\omega_{n_l})\big\| + \big\|\Gamma^{k-1}(\omega_{n_l}) - u\big\|.$$

Using the above inequality and Lemma 5, we get

$$\liminf_{l\to\infty}\big\|\Gamma^{k-1}(\omega_{n_l}) - \Gamma(u)\big\| \leq \liminf_{l\to\infty}\big\|\Gamma^{k-1}(\omega_{n_l}) - u\big\|,$$

which contradicts the Opial condition. Therefore, u must be a fixed point of Γ.

Next, we claim that the sequence $\{\Gamma^{k-1}(\omega_n)\}$ converges weakly to u. Assume, to the contrary, that a subsequence $\{\Gamma^{k-1}(\omega_{m_j})\}$ converges weakly to some $v \in W$ with $u \neq v$. Using a similar argument, one can establish that v is also a fixed point of Γ.

Since both $\lim_{n\to\infty} \|\Gamma^{k-1}(\omega_n) - u\|$ and $\lim_{n\to\infty} \|\Gamma^{k-1}(\omega_n) - v\|$ exist, the Opial condition ensures that

$$\begin{aligned}
\lim_{n\to\infty}\big\|\Gamma^{k-1}(\omega_n) - u\big\| &= \liminf_{l\to\infty}\big\|\Gamma^{k-1}(\omega_{n_l}) - u\big\| \\
&< \liminf_{l\to\infty}\big\|\Gamma^{k-1}(\omega_{n_l}) - v\big\| \\
&= \liminf_{j\to\infty}\big\|\Gamma^{k-1}(\omega_{m_j}) - u\big\| \\
&< \liminf_{j\to\infty}\big\|\Gamma^{k-1}(\omega_{m_j}) - v\big\| \\
&= \lim_{n\to\infty}\big\|\Gamma^{k-1}(\omega_n) - u\big\|,
\end{aligned}$$

which is a contradiction.

Thus, the sequence $\{\Gamma^{k-1}(\omega_n)\}$ converges weakly to $u \in W$. $\square$

Now, we establish some fixed point results for k-nonexpansive mappings in Banach spaces with normal structure.

Theorem 3. *Let W be a nonempty, convex, weakly compact, and bounded subset of a Banach space Y with normal structure. If $\Gamma : W \to W$ is a k-nonexpansive mapping, then Γ admits a fixed point.*

Proof. Since W is a weakly compact subset of a Banach space Y with normal structure, Zorn's Lemma ensures the existence of a nonempty, closed, convex, and Γ-invariant subset $U \subseteq W$ that contains no proper subsets with these properties.

Fix an arbitrary point $\omega_0 \in U$, and consider the sequence $\{\Gamma^n(\omega_0)\}$, which is bounded in U. By the normal structure of Y, it follows from [1, Corollary 1] that the function $g : U \to [0, \infty)$, defined as

$$g(\omega) := \limsup_{n\to\infty} \big\|\omega - \Gamma^n(\omega_0)\big\|,$$

is not constant and remains bounded on the convex hull $\operatorname{conv}\{\Gamma^n(\omega_0) : n \in \mathbb{N}\}$. Consequently, there exist points $u_1, u_2 \in U$ such that

$$r_1 := g(u_1) < g(u_2) =: r_2.$$

Set $r = \frac{1}{2}(r_1 + r_2)$, and define the set

$$M := \big\{\omega \in U : g(\omega) \leq r\big\}.$$

It is straightforward to verify that M is a nonempty, closed, convex subset of U, and $M \neq U$ since $u_2 \notin M$.

Next, taking $\omega = \Gamma^n(\omega_0)$ and any $\vartheta \in U$, we apply the k-nonexpansive condition:

$$\big\|\Gamma^{n+k}(\omega_0) - \Gamma(\vartheta)\big\| \leq \big\|\Gamma^{n+k-1}(\omega_0) - \vartheta\big\|.$$

Taking the $\limsup$ on both sides, we get

$$g(\Gamma(\vartheta)) \leq g(\vartheta) \leq r.$$

This implies that M is a nonempty, closed, convex, Γ-invariant subset of U, contradicting the minimality of U.

Therefore, Γ has a fixed point in W. □

Corollary 1. *Let W be a nonempty, weakly compact convex subset of a UCED Banach space Y. If $\Gamma : W \to W$ is a k-nonexpansive mapping, then Γ has a fixed point.*

Corollary 2. *Let W be a nonempty, closed bounded, and convex subset of a uniformly convex Banach space Y. If $\Gamma : W \to W$ is k-nonexpansive mapping then Γ has a fixed point.*

Corollary 3. *Let W be a nonempty, closed bounded, and convex subset of a real Hilbert space Y. If $\Gamma : W \to W$ is k-nonexpansive mapping then Γ has a fixed point.*

5. Conclusion

In this paper, we have introduced the concept of k-nonexpansive mappings, a significant generalization of fundamentally nonexpansive (FNE) mappings. This new class of mappings contains various mappings such as nonexpansive mappings, FNE mappings, and mappings satisfying the condition (C). We established that the fixed point set of a k-nonexpansive mapping, defined on a nonempty convex subset of a strictly convex normed space, is convex. Some results that ensure the existence of an afps for k-nonexpansive mappings are presented under specific conditions on the range of the mappings. Notably, we proved some fixed point theorems for k-nonexpansive mappings in the setting of Banach spaces with either normal structure or the Opial property, which provide a partial answer to the question raised by Domínguez Benavides and Llorens-Fuster regarding FNE mappings.

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